THE GOING UP AND LYING OVER THEOREMS IN BL-ALGEBRAS

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ABSTRACT. In this paper, by considering the notion of residuated lattice and BL algebra, we concern a relationship between the important theorems in commutative algebra and BL-algebra theory.

1. INTRODUCTION AND PRELIMINARIES

The origin of BL-algebras is in Mathematical Logic; they were invented by Hájek in [2]. Apart from their logical interest, BL-algebras have important algebraic properties (see [2]). MV-algebras are BL-algebras with double negation. Authors have been working on MV-algebras in [3, 4].

Definition 1.1. A residuated lattice \( A = (A, \lor, \land, \odot, \rightarrow, 1) \) is a lattice \( A \) containing the largest element 1, and endowed with two binary operations \( \odot \) (called product) and \( \rightarrow \) (called residuum) satisfying the following:

\[
\begin{align*}
(A_{\land,1}) & \quad (A, \odot, 1) \text{ is a commutative po-monoid}, \\
(AP) & \quad \odot \text{ and } \rightarrow \text{ form an adjoint pair, i.e. } x \odot y \leq z \text{ if and only if } y \leq x \rightarrow z, \text{ for all } x, y, z \in A. \quad \text{Furthermore, } A \text{ is called a BL-algebra if it satisfies the following:} \\
(div) & \quad x \land y = x \odot (x \rightarrow y). \\
(pre) & \quad (x \rightarrow y) \lor (y \rightarrow x) = 1.
\end{align*}
\]

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A bounded residuated lattice is a residuated lattice \((A, \lor, \land, \circ, \rightarrow, 0, 1)\) in which \((A; \lor, \land, 0, 1)\) is a bounded lattice. In a bounded residuated lattice we can define a negation, \(\ast\), by: \(x^* := x \rightarrow 0\), for all \(x \in A\). For any bounded residuated lattice \(A\) we denote \((x^*)^*\) by \(x^{**}\).

The following three structures are main examples of BL-algebras on real unit interval.

**Example 1.2.**

(i) Gödel structure: \(x \odot_G y := \min\{x, y\}\) and \(x \rightarrow_G y := \begin{cases} 1 & \text{if } x \leq y, \\ y & \text{otherwise}. \end{cases}\)

(ii) Product structure: \(x \odot_P y := xy\) and \(x \rightarrow_P y := \begin{cases} 1 & \text{if } x \leq y, \\ y & \text{otherwise}. \end{cases}\)

(iii) Lukasiewicz structure: \(x \odot_L y := \max(x + y - 1, 0)\) and \(x \rightarrow_L y := \min(y - x + 1, 1)\).

(iv) If \((A, \oplus, *, 0)\) is an MV-algebra, then \((A, \lor, \land, \odot, \rightarrow, 0, 1)\) is a BL-algebra, where for \(x, y \in A\):

\[
x \odot y := (x^* \oplus y^*)^*, \quad x \rightarrow y := (x \odot y^*)^*, \quad x \lor y := (x^* \lor y^*)^* \quad \text{and} \quad 1 := 0^*.
\]

In order to define additivity we define by the de Morgan role a new binary operation “\(\oplus\)” by the formula

\[
x \oplus y := (x^* \odot y^*)^*.
\]

**Proposition 1.3.** *Let \(A\) be a BL-algebra which the arbitrary meets and unions exist. Then the following conditions satisfy for each \(x, y, z \in A\).*

\[
\begin{align*}
(c_1) \quad & x \oplus x^* = 1, \\
(c_2) \quad & x \leq (x \odot y) \oplus y^*, \\
(c_3) \quad & x \lor y \leq x \oplus y, \\
(c_4) \quad & x \oplus 0 = x^{**} \quad \text{and} \quad x \oplus 1 = 1.
\end{align*}
\]

Let \(A\) be a BL-algebra and \(F\) be a nonempty subset of \(A\). \(F\) is called a filter of \(A\) if satisfies the following conditions for all \(x, y \in A\): (fi1) \(x, y \in F\) implies \(x \odot y \in F\), (fi2) \(x \leq y\) and \(x \in F\) imply \(y \in F\). A filter \(F\) of \(A\) is proper if and only if \(F \neq A\). A proper filter \(P\) of \(A\) is called prime provided that \(x \lor y \in P\) implies \(x \in P\) or \(y \in P\). Let \(A\) be a bounded residuated lattice and \(F\) be a filter of \(A\). Then we say \(F\) is a Boolean filter if, for all \(x \in A\), we have \(x \lor x^* \in F\). Boolean filters of BL-algebra \(A\) is denoted by \(BF(A)\) and \(Spec(A) \cap BF(A)\) is
denoted by $BFS(A)$.

**Proposition 1.4.** Let $A$ be a BL-algebra and $F, G$ be two filters of $A$ such that $F \subseteq G$. Then $x/F \in G/F$ implies that $x \in G$. Furthermore, if $G/F$ is a filter of the quotient BL-algebra $A/F$, then $G \in Spec(A)$ if and only if $G/F \in Spec(A/F)$.

In the following, we generalize the Chinese remainder theorem for BL-algebras.

**Theorem 1.5.** Let $F_1, \cdots, F_n$ be filters of a BL-algebra $A$ such that $F_i$ and $F_j$, for $i \neq j$, $i, j \in \{1, \cdots, n\}$, are comaximal. Then for each $x_1, \cdots, x_n \in A$, there is an $x \in A$ such that $x \equiv_{F_i} x_i^*$.

**Corollary 1.6.** Let $A$ be a BL-algebra and $F_1, \cdots, F_n$ be filters of a BL-algebra $A$. Then there exists an injective BL-morphism $\Theta : A/(\cap_{i=1}^n F_i) \rightarrow A/F_1 \times \cdots \times A/F_n$. If $F_i \odot F_j = A$ for $i \neq j$, $i, j \in \{1, \cdots, n\}$, then for each $(a_1/F_1, \cdots, a_n/F_n)$ there exists $a \in A$ such that $\Theta(a/(\cap_{i=1}^n F_i)) = (a_1^*/F_1, \cdots, a_n^*/F_n)$.

**Proposition 1.7.** Let $A$ and $A'$ be two BL-algebras and $f : A \rightarrow A'$ be a BL-morphism. Then $\tilde{f} : Spec(A') \rightarrow Spec(A)$, is defined by $\tilde{f}(P) = f^{-1}(P)$ for each $P \in Spec(A')$, is a continuous map with respect to Zariski topology.

2. **Main results**

In this section, first by considering the notion of BL-algebras we state and prove the going up and lying over theorems version for BL-algebras. In fact, Belluce, in [1], proved the going up and lying over theorem for MV-algebras and since in MV-algebras there is a nice symmetry between $1, \oplus$, sup and $0, \odot$, inf, respectively, these theorems were proved in a good manner. Propositions 1 to 5 in [1] are aimed to prove the localization theorem for MV-algebras, but in BL framework this main result is missing. By any way, we prove that if $S$ is a subalgebra of $A$ and $P \in BFS(S)$, then there exists $Q \in Spec(A)$ such that $P = Q \cap S$. In what follows, $A$ and $A'$ will denote BL-algebras and $f : A/\alpha A'$ an BL-morphism. $\tilde{f}$ will denote the induced continuous map, $\tilde{f} : Spec(A') \cap Spec(A)$, with $\tilde{f}(P) = f^{-1}(P)$ where $P \in spec(A')$. The
following two theorems are almost direct generalization of Proposition 1 and 2 in [1].

Definition 2.1. We say the Going up theorem holds for $f$ provided for any $P \in \text{Spec}(A)$ and any Boolean prime filter $Q$, $P \subseteq Q$, if there exists $P' \in \text{Spec}(A')$ such that $\tilde{f}(P') = P$, then there exists $Q' \in \text{Spec}(A')$ such that $P' \subseteq Q'$ and $\tilde{f}(Q') = Q$.

Theorem 2.2. (Going up theorem) If $P \subseteq Q$, $P \in \text{Spec}(A)$, $Q \in BF(A)$ and $\tilde{f}(P') = P$ for a prime filter $P' \in \text{Spec}(A')$, then there exists $Q' \in \text{Spec}(A')$ such that $P' \subseteq Q'$ and $\tilde{f}(Q') = Q$.

Definition 2.3. Let $A$ and $A'$ be two BL-algebras and $f : A \longrightarrow A'$ be a BL-morphism. We say $f$ satisfies the Lying over theorem (LO) provided for each $P \in BFS(A)$ with $\ker(f) \subseteq P$ there is a $P' \in \text{Spec}(A')$ such that $\tilde{f}(P') = P$.

Theorem 2.4. (Lying over theorem) Let $A$ and $A'$ be two BL-algebras. Then each BL-morphism $f : A \longrightarrow A'$ satisfies the lying over theorem.

Corollary 2.5. Let $A$ and $A'$ be two BL-algebras and $f : A \longrightarrow A'$ be a BL-morphism. If $f$ is injective, then $BFS(A) \subseteq \text{Im}(\tilde{f})$.

Corollary 2.6. If $S$ is a subalgebra of $A$ and $P \in BFS(S)$, then there exists a $Q \in \text{Spec}(A)$ such that $P = Q \cap S$.

REFERENCES


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