SOME CODES AND DESIGNS INVARIANT UNDER $PSU_3(3)$

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Abstract. We construct some designs and associated binary codes from a primitive permutation representation of the simple group $PSU_3(3)$ and we establish some properties of these designs and codes.

1. Introduction and Preliminaries

In [4, 5, 6, 7], some designs and codes have been defined by the action of the simple group on the cosets of its maximal subgroups over $GF(2)$. Also, in the mentioned papers, the full automorphism groups of those designs and codes have been studied. In this talk, our notation will be standard and for the structure of groups and their maximal subgroups we follow the ATLAS notation. The groups $G.H$ and $G : H$ denote a non-split extension and split extension, respectively. Let $S = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ be a finite incidence structure which consists of two disjoint finite sets $\mathcal{P}$ and $\mathcal{B}$, and a subset $\mathcal{I}$ of $\mathcal{P} \times \mathcal{B}$. The members of $\mathcal{P}$ and $\mathcal{B}$ are called points and blocks, respectively. For $p \in \mathcal{P}$ and $B \in \mathcal{B}$, we will write $p \mathcal{I} B$ if and only if $(p, B) \in \mathcal{I}$. The incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ is called a $t - (\nu, k, \lambda)$ design if $|\mathcal{P}| = \nu$, $|B| = k$ for each $B \in \mathcal{B}$ and every $t$ points of $\mathcal{P}$ is incident with precisely $\lambda$ blocks of $\mathcal{B}$. The design $\mathcal{D}$ is called symmetric if the number of points $\nu$ is equal to the number of blocks $b$. In a 2-design with $k < \nu$, the following conditions are equivalent:

(a) $b = \nu$;
(b) $r = k$;

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(c) any two blocks have $\lambda$ common points;
(d) every two blocks have a constant number of the number of common point.

A 2-design is called square if it satisfies the equivalent conditions. A steiner system $S(t, k; \nu)$ is a $t-(\nu, k, 1)$ design. An incidence matrix of a structure $D = (P, B, I)$ is a matrix $A$ whose rows and columns are indexed by the blocks and points of the designs respectively, the entry indexed by $(B, P)$ being $1$ if $P \in B$, $0$ otherwise. The code $C_F$ of the design $D$ over the finite field $F$ is the space spanned by the incidence vectors of the blocks over $F$. We take $F$ to be a prime field $GF(p)$, in which case we write also $C_p$ for $C_F$. Thus $C_F = \langle \nu^B | B \in B \rangle$, and is a subspace of $F^P$, the full vector space of functions from $P$ to $F$. If a linear code over a field of order $q$ is of length $n$, dimension $k$ and minimum distance $d$, then we write $[n, k, d]_q$ to represent this information. Elements of $C$ are called codewords and the support of $c \in C$ is the set of non-zero coordinate positions of $c$. A binary code, that is a code over $GF(2)$, with all weights divisible by 2 or 4 is called even or doubly even, respectively.

Terminology for graphs is standard [2]: A graph consists of a finite set of vertices and a set of edges, with an incidence relation between vertices and edges having the properties that any edge is incident with exactly two vertices, and any two vertices are incident with at most one edge. Our graphs are undirected (we do not allow edges to be ordered pairs), the valency of a vertex is the number of edges containing the vertex. A graph is regular if all the vertices have the same valence, and a regular graph is strongly regular of type $(n, k, \lambda, \mu)$ if it has $n$ vertices, valence $k$, and if any two adjacent vertices are together adjacent to $\lambda$ vertices, while any two non-adjacent vertices are together adjacent to $\mu$ vertices.

**Proposition 1.1.** [3, 6] Let $G$ be a finite primitive permutation group acting on the set $\Omega$ of size $n$. Let $\alpha \in \Omega$ and let $\Delta \neq \{\alpha\}$ be an orbit of the stabilizer $G_\alpha$ of $\alpha$. If

$$B = \{\Delta^g : g \in G\}$$

$$\varepsilon = \{\{\alpha, \delta\}^g : g \in G\},$$

for a given $\delta \in \Delta$, then the incidence structure $D = (\Omega, B)$ forms a symmetric $1 - (n, |\Delta|, |\Delta|)$ design. Further, if $\Delta$ is a self-paired orbit of $G_\alpha$, then $\Gamma = (\Omega, \varepsilon)$ is a regular connected graph of valency $|\Delta|$, $D$ is self-dual and $G$ acts as
an automorphism group on each of these structures, primitive on vertices of
the graph and on points and blocks of the design.

**Lemma 1.2.** [8]

(i) Let $\mathcal{D}$ be a self-dual 1-design obtained by taking all the images under
the group $G$ of a non-trivial orbit of the point stabilizer in any of $G$’s
primitive representations, and on which $G$ acts primitively on points and
blocks, then $G \leq \text{Aut}(\mathcal{D})$.

(ii) If $\mathcal{C}$ is a linear code of length $n$ of a symmetric $1 - (\nu, k, k)$ design $\mathcal{D}$
over a finite field $F_q$, then $\text{Aut}(\mathcal{D}) \leq \text{Aut}(\mathcal{C})$.

2. Main results

By [3], $PSU_3(3)$ has four maximal subgroups up to conjugacy of orders 168,
96, 96, 216. For the last subgroup, we have $\sharp = 2$ and for this subgroup, the
design is trivial. Thus we consider the designs and codes for the first three
subgroups:

**Theorem 2.1.** (i) For $PSU_3(3)$ of degree 36, the two obtained $1 - (36, 7, 7)$
designs are isomorphic and thus we denote them by $\mathcal{D}_7$. Moreover,
$\text{Aut}(\mathcal{D}_7) \cong PSU_3(3)$.

(ii) For $PSU_3(3)$ of degree 36, the automorphism group of the $1 - (36, 21, 21)$
design $\mathcal{D}_{21}$ is isomorphic to the $PSU_3(3) : 2$.

**Note 2.2.** For $PSU_3(3)$ of degree 36, the $1 - (36, 21, 21)$ design $\mathcal{D}_{21}$ is actually
a square $2 - (35, 21, 12)$ design.

**Theorem 2.3.** For $PSU_3(3)$ of degree 63, the full automorphism groups of the
$\mathcal{D}_6$, $\mathcal{D}_{24}$ and $\mathcal{D}_{32}$ are isomorphic to the $PSU_3(3) : 2$.

**Note 2.4.** For $PSU_3(3)$ of degree 63, the $1 - (63, 32, 32)$ design $\mathcal{D}_{32}$ is actually
a square $2 - (63, 32, 64)$ design.

**Theorem 2.5.** (i) For $PSU_3(3)$ of degree 63, the two obtained $1 - (63, 16, 16)$
designs are isomorphic and hence, we denote them by $\mathcal{D}_{16}$. Moreover,
$\text{Aut}(\mathcal{D}_{16}) \cong PSU_3(3)$;

(ii) For $PSU_3(3)$ of degree 63, the automorphism group of the $1 - (63, 6, 6)$
design $\mathcal{D}_6$ and $1 - (63, 24, 24)$ design $\mathcal{D}_{24}'$ are isomorphic to the $PSU_3(3) : 2$. 
Note that all of these automorphism groups of the designs by same order are isomorphic. Then we have \( \text{Aut}(\mathcal{D}_7) \cong \text{Aut}(\mathcal{D}_{16}) \cong \text{PSU}_3(3) \) and for every \( i, j \in \{6, 21, 24, 32\} \), \( \text{Aut}(\mathcal{D}_i) = \text{Aut}(\mathcal{D}_j) \cong \text{PSU}_3(3) : 2 \)

**Remark 2.6.** For every \( i \in \{24, 32\} \), \( \text{Aut}(\mathcal{D}_i) \cong \text{Aut}(\mathcal{C}_i) \cong \text{PSU}_3(3) : 2 \).

**Result 2.7.** For every \( i \in \{6, 21, 24, 32\} \) and \( j \in \{16, 24, 32\} \), \( \text{Aut}(\mathcal{D}_i) \cong \text{Aut}(\mathcal{C}_j) \cong \text{PSU}_3(3) : 2 \).

**References**


